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A GENERAL ALGEBRAIC METHOD FOR THE SOLUTION OF EQUATIONS.

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IN Algebra, as now taught, equations of the second, third and fourth degrees, are resolved, each by a specific method or artifice specially adapted to the degree of the equation. It may not be an inappropriate addition to supplement these diverse methods by an uniform general method, applicable independently of the degree of the proposed equation.

Space will permit only of a condensed statement of the principles and outlines of the proposed method.

I.

Each root of an equation of the nth degree, wanting its second term, is equal to the sum of the roots of $n-1$ binomial equations of the form $y^n = a^n$, $y^n = b^n$, $y^n = c^n$, &c., one root being taken from each binomial equation in the formation of each root of the original equation.

II.

The order in which the n roots of each of the $n-1$ binomial equations are arranged to form by their sum the n roots of the original equation is symmetrical, and is given in the following table, in which ε represents any one of the imaginary nth roots of unity:

$$\underline{x = \sqrt[n]{a^n} + \sqrt[n]{b^n} + \sqrt[n]{c^n} + \sqrt[n]{d^n} + \ldots + \text{to } n - 1 \text{ terms.}}$$

$$\begin{array}{l}
x = a(\varepsilon^n) + b(\varepsilon^n)^2 + c(\varepsilon^n)^3 + d(\varepsilon^n)^4 + \ldots + m(\varepsilon^n)^{n-1}, \\
x = a(\varepsilon\ ) + b(\varepsilon^2) + c(\varepsilon^3) + d(\varepsilon^4) + \ldots + m(\varepsilon^{n-1}), \\
x = a(\varepsilon^2) + b(\varepsilon^2)^2 + c(\varepsilon^2)^3 + d(\varepsilon^2)^4 + \ldots + m(\varepsilon^2)^{n-1}, \\
x = a(\varepsilon^3) + b(\varepsilon^3)^2 + c(\varepsilon^3)^3 + d(\varepsilon^3)^4 + \ldots + m(\varepsilon^3)^{n-1}, \\
x = \qquad + \qquad + \qquad + \qquad + \qquad + \qquad , \\
x = a(\varepsilon^{n-1}) + b(\varepsilon^{n-1})^2 + c(\varepsilon^{n-1})^3 + d(\varepsilon^{n-1})^4 + \ldots + m(\varepsilon^{n-1})^{n-1}.
\end{array}$$

It will be noticed that in the first perpendicular column a is combined with *all* the nth roots of unity, b in the second column with the squares of these roots, c in the third column with their cubes &c. But as obviously $\varepsilon^n = 1$, $\varepsilon^{n+1} = \varepsilon$, $\varepsilon^{n+2} = \varepsilon^2$, &c., each letter in the table is combined in each instance with an nth root of unity, and this combination is therefore always one of the nth roots of the binomial equation in that letter.

III.

If n values of x be taken from the above table and combined to form an equation of the nth degree, the imaginary quantities ε, ε^2, ε^3, &c., will wholly *disappear* and the coefficients and the absolute term of the equation will all be rational functions of the $n—1$ quantities a, b, c, &c.

If numerical values be assigned to the letters a, b, c, &c., the equation produced, of whatever degree, will verify all the values of x, and its coefficients and the absolute term will all be real numbers.

IV.

When n is a prime number, the coefficients and the absolute term of the equation produced are symmetrical rational functions of the $n—1$ quantities a, b, c, &c.; symmetrical in the important sense that each coefficient and the absolute term is exactly the same function of one letter that it is of another, and the same interchange between the letters is permissible as when n letters are used to represent the n values of the unknown quantity.

When n is a composite number, while the coefficients are no longer symmetrical functions of a, b, c, &c , the relations established between these quantities afford facilities for the reduction of the equation.

V.

The resolution of a given equation is effected by giving expression to its coefficients and the absolute term as functions of the n-1 letters a, b, c, &c., and then, by eliminating from the supplemental equations thus formed all the letters but one, producing an auxiliary equation of the $(n—1)$th degree, whose resolution affords the elements forming by their sum the values of the unknown quantity in the given equation.

Applying this method to the general equations of the several degrees in their order, we have—

VI.

QUADRATIC EQUATIONS.

The artifice usually employed is termed "completing the square" by an addition to each member of the equation. While this is effective and the most available for practical use, the general method is, under its terms equally applicable.

Given the general quadratic equation

$$x^2 + 2px = q;$$

removing the second term we have the equation $y^2 = p^2 + q$. As $n = 2$ and $n—1 = 1$, only one letter, a, can be employed, and this letter is to be combined with the two square roots of unity to form the two values of y.

$$\begin{array}{ll} y = a(\ 1) = \ \ \sqrt{(p^2 + q)} & x = -p + \sqrt{(p^2 + q)}, \\ y = a(-1) = -\sqrt{(p^2 + q)} & x = -p - \sqrt{(p^2 + q)}. \end{array}$$

VII.

CUBIC EQUATIONS.

Here the general method corresponds closely with the particular method usually employed.

Given the general cubic equation

$$x^3 + 3px = 2q;$$

we form from the above table the following:

$$\begin{array}{l} \underline{x = \sqrt[3]{a^3} + \sqrt[3]{b^3}.} \\ x = a + b \\ x = a\varepsilon + b\varepsilon^2 \\ x = a\varepsilon^2 + b\varepsilon. \end{array}$$

From these three values of x we construct the general equation

$$x^3 - 3abx = a^3 + b^3,$$

whose coefficient and absolute term are symmetrical functions of a and b. The supplemental equations become $ab = -p$ and $a^3 + b^3 = 2q$.

Eliminating b the final equation is a quadratic. But as $\sqrt[3]{a^3}$ and $\sqrt[3]{b^3}$ have each three values, these six unknown quantities, expressed in the table, must all enter into the final equation of which they will be the roots; this equation will, therefore, be not only a quadratic, but it must also be an equation of the 6th degree, to wit:

$$a^6 - 2qa^3 = p^3,$$

whence

$$a^3 = q + \sqrt{(q^2 + p^3)}, \text{ and } b^3 = q - \sqrt{(q^2 + p^3)},$$

and

$$x = \sqrt[3]{[q + \sqrt{(q^2 + p^3)}]} + \sqrt[3]{[q - \sqrt{(q^2 + p^3)}]}.$$

It may be observed that the interchange of a and b with each other in the equation $x^3 - 3abx = a^3 + b^3$ does not change the value of the roots, its only effect being to change the order of the radicals in the above expression of the value of x.

The essential reason for the existence of the "irreducible case" is also here made manifest.

The values of x expressed in the foregoing table are one real and two imaginary; an attempt therefore to combine them to form a cubic equation whose roots are all real must necessarily result in the imposition of impossible conditions in the final formula.

VIII.

BIQUADRATIC EQUATIONS.

Instead of resorting to the variety of expedients heretofore employed, apply the above method directly to the general equation

$$x^4 - 2px^2 - 8qx = r.$$

The four 4th roots of unity are 1, -1, $\sqrt{-1}$ and $-\sqrt{-1}$. Applying these to the general table we have:—

$$\begin{aligned} x &= \quad a + b + c \\ x &= -a + b - c \\ x &= \quad a\sqrt{-1} - b - c\sqrt{-1} \\ x &= -a\sqrt{-1} - b + c\sqrt{-1}. \end{aligned}$$

Here x in general $= \sqrt[4]{a^4} + \sqrt{b^2} + \sqrt[4]{c^4}$, but as the square roots of b^2 are 4th roots of b^4 the conditions of sections I and II are still observed.

Constructing an equation from the four values of x in the table we have:

$$x^4 - 2(b^2 + 2ac)x^2 - 4b(a^2 + c^2)x = a^4 - b^4 + c^4 - 2a^2c^2 + 4ab^2c;$$

giving the supplemental equations

$$\begin{aligned} b^2 + 2ac &= p, \\ b(a^2 + c^2) &= 2q, \\ a^4 - b^4 + c^4 - 2a^2c^2 + 4ab^2c &= r. \end{aligned}$$

As 4 is not a prime number, there is here an absence of symmetry; the relations established, however, in the supplemental equations, enable us to ascertain with great facility the values of a, b and c.

For brevity take first the simplest general form

$$x^4 - 8qx = r,$$

and hence $p = 0$, $-2ac = b^2$ and $2q \div (a^2 + c^2) = b$. Substituting these values there is produced the auxiliary cubic equation

$$b^6 + \tfrac{1}{4}rb^2 = q^2,$$

whence the value of b is readily obtained and the values of a and c also determined, when the four values of x may be expressed as indicated in the table. When p does not equal 0 the auxiliary cubic equation is found, but with a second term in b^4.

As in the forgoing table the values of x are two real and two imaginary, an "irreducible case" must arise whenever the roots are all real or all imaginary, and this essential reason is here first disclosed, without reference to the character of the auxiliary cubic equation.

IX.

EQUATIONS OF THE 5th DEGREE.

In entering upon this almost *terra incognita*, we have in the proposed general method a simple and direct line of attack upon the difficulties of the case.

Representing by ε any one of the imaginary 5th roots of unity we have from the general table:—

$$\begin{array}{l}
\underline{x = \sqrt[5]{a} + \sqrt[5]{b} + \sqrt[5]{c} + \sqrt[5]{d}.} \\
x = a \quad + b \quad + c \quad + d, \\
x = a\varepsilon + b\varepsilon^2 + c\varepsilon^3 + d\varepsilon^4, \\
x = a\varepsilon^2 + b\varepsilon^4 + c\varepsilon + d\varepsilon^3, \\
x = a\varepsilon^3 + b\varepsilon + c\varepsilon^4 + d\varepsilon^2, \\
x = a\varepsilon^4 + b\varepsilon^3 + c\varepsilon^2 + d\varepsilon.
\end{array}$$

We then construct, directly, an equation of the 5th degree, having the above five values of x for its roots.

Again the imaginary quantities wholly disappear, and we have the eq'n

$$x^5 - 5(ad+bc)x^3 - 5(a^2c+ab^2+bd^2+c^2d)x^2 + 5(a^2d^2 - a^3b - abcd - ac^3 - b^3d - cd^3 + b^2c^2)x = a^5+b^5+c^5+d^5 - 5(a^3cd - a^2b^2d - a^2bc^2 + ab^3c + abd^3 - ac^2d^2 - b^2cd^2 + bc^3d).$$

Given the general equation

$$x^5 + 5px^3 + 5qx^2 + 5rx = s,$$

we have the supplemental equations

$$ad + bc = -p,$$
$$a^2c + ab^2 + bd^2 + c^2d = -q,$$
$$a^2d^2 - a^3b - abcd - ac^3 - b^3d - cd^3 + b^2c^2 = r, \text{ and}$$
$$a^5 + b^5 + c^5 + d^5 - 5(a^3cd - a^2b^2d - a^2bc^2 + ab^3c + abd^3 - ac^2d^2 - b^2cd^2 + bc^3d) = s.$$

Since the above supplemental equations are each exactly the same function of one letter as of another, and as both in them and in the table one letter may be interchanged with another without disturbing any relation, the final equation obtained by elimination will be of one identical form, whichever three of the four unknown quantities are eliminated.

This final equation will, therefore, contain within it, as its roots, all the values of each of the four letters a, b, c and d, which values are expressed in the above table, and it can contain no other values of the unknown quantity.

Were it not for the permutations which must take place, this final auxiliary equation would be an equation of the fourth degree, of the form

$$a^{20} - Aa^{15} + Ba^{10} - Ca^5 + D = 0.$$

This is readily made apparent. By reference to the above table it will be observed that twenty distinct quantities a, $a\varepsilon^2$, $b\varepsilon^4$, &c, all unknown, enter into the composition of the coefficients of the given general equation. If any one of the five values of a in the table be assigned to a in the supplemental equations, this value of a, together with the corresponding values of b, c and d in the same horizontal column, or value of x, will satisfy all the equations, and the same is true of the five values of b, c and d respectively. The table also embraces all the possible values of a, b, c, and d, as it is impossible to assign any other value and verify the equations, all of which is in strict analogy with the same relations in the table ot the cubic eq'n.

Consequently the final equation, in which there is centered in one letter, by elimination, all the values of the four letters, must have for its roots, did permutation not take place, precisely these twenty quantities contained in the table, in strict analogy to the final auxillary equation containing the six quantities of the cubic table.

But the equation having for its roots these twenty quantities (as is made manifest by its formation directly from these roots), can only be of the form:

$$y^{20} - (a^5 + b^5 + c^5 + d^5)y^{15} + (a^5b^5 + a^5c^5 + a^5d^5 + b^5c^5 + b^5d^5 + c^5d^5)y^{10}$$
$$- (a^5b^5c^5 + a^5b^5d^5 + a^5c^5d^5 + b^5c^5d^5)y^5 + a^5b^5c^5d^5 = 0,$$

which is the form of the auxiliary equation above indicated.

But permutations take place. The four perpendicular rows in the table in a, b, c and d respectively are capable of $1 . 2 . 3 . 4 = 24$ permutations with each other. Consequently the final equation in a will not be of the 20th, but of the 120th degree. But by these permutations no unknown quantities additional to the twenty in the table are introduced into the final equation, and each of these twenty quantities enters into that equation precisely the same number of times. The entire twenty quantities are therefore included six times in the formation of the equation of the 120th degree. It is a necessary conclusion that the first member of this equation, the second member being zero, is a polynomial which is a perfect sixth power, and that the sixth root of this polynomial is the first member of the required equation in a^{20}, whose roots are the twenty quantities expressed in the table. As this equation is a biquadratic, its solution is practicable, and its roots may be combined, as in the table, to form the five values of x in the given general equation of the fifth degree.

It will be observed that in the above table the values of x are one real and four imaginary, consequently when all the roots are real, or when only two are imaginary, an "irreducible case" will again be presented.

X

It is a remarkable fact, in this connection, that, by reason of the radicals therein, the algebraic expression for the value of a in the foregoing equation

$$a^{20} - Aa^{15} + Ba^{10} - Ca^5 + D = 0,$$

is capable of expressing algebraically 120 values of a, while it only expresses 20 actually different values.

For convenience transform the equation into one wanting its second term:

$$a^{20} + 2pa^{10} + 8qa^5 = r.$$

Let

$$a^5 = \sqrt{n} + \sqrt{-\left(p+n+\frac{2q}{\sqrt{n}}\right)};$$

make the substitution and we have the cubic equation

$$n^3 + pn^2 + \frac{p^2 + r}{4} n = q^2,$$

and the three values of n may be each expressed in the regular cubic formula.

The four values of a^5 in the above equation are expressed as follows:

$$a^5 = \sqrt{n} + \sqrt{-\left(p+n+\frac{2q}{\sqrt{n}}\right)},$$

$$a^5 = \sqrt{n} - \sqrt{-\left(p+n+\frac{2q}{\sqrt{n}}\right)},$$

$$a^5 = -\sqrt{n} + \sqrt{-\left(p+n-\frac{2q}{\sqrt{n}}\right)},$$

$$a^5 = -\sqrt{n} - \sqrt{-\left(p+n-\frac{2q}{\sqrt{n}}\right)}.$$

But here are four different algebraic expressions for the value of a^5 containing the symbol n, as yet unexpressed in its cubic formula. As n, however, is capable of expression in six different algebraic forms, these values of n successively substituted in each of the above four expressions will give twenty four different algebraic expressions for the value of a^5.

(To obtain the six algebraic forms expressing the value of n, for convenience, transform the cubic equation into the well known general form

$$x^3 + 3px = 2y;$$

let

$$x = \sqrt[3]{y} - \frac{p}{\sqrt[3]{y}}$$

and make the substitution, resulting in the quadratic

$$y^2 - 2qy = p^3,$$

in which y has the two values $q + \sqrt{(q^2 + p^3)}$ and $q - \sqrt{(q^2 + p^3)}$.

But the three values of x are expressed as follows, representing by ε one of the imaginary cubic roots of unity:

$$x = \sqrt[3]{y} - \frac{p}{\sqrt[3]{y}},$$

$$x = \varepsilon \sqrt[3]{y} - \frac{p}{\varepsilon \sqrt[3]{y}},$$

$$x = \varepsilon^2 \sqrt[3]{y} - \frac{p}{\varepsilon^2 \sqrt[3]{y}}.$$

Substituting for y in the above formulæ each of its two values we have the six different algebraic expressions for the value of x in the cubic equation.)

As a is the 5th root of each of the 24 algebraic expressions for the value of a^5, and as there are five 5th roots in each case, expressed by combination with the one real and four imaginary 5th roots of unity, there are therefore 120 different algebraic expressions for the value of a in the foregoing equation.

These 120 varied expressions are the necessary result of the permutations which have taken place, but upon analysis they are found to represent only twenty different quantities, and these are the quantities found in the table.

[We feel much interest in the foregoing paper and invite a critical examination of it by our readers.

We have never devoted much attention to the efforts that have been made to solve the general equation of the fifth degree, and hence do not regard ourself as a competent critic; and as the extended efforts of Hirsch, Schulenburg and others, have resulted in failures, and especially as so exalted a mathematician as Abel has pronounced the solution impossible, it would seem presumptious to assert that the problem has been solved in the foregoing paper. Nevertheless, as we have not been able to detect any error in the foregoing very elegant discussion of the question, we earnestly hope that, if it shall not be found in every respect complete, it may lead to the desired solution, and we commend it to the careful consideration and critical scrutiny of our readers.—Ed.]